Iteration

There are some equations that cannot be solved using 'standard' methods to find an exact solution, but an approximate solution can usually be found using a numerical method based on iteration.

Bisection

The bisection method is based on finding two values between which the solution lies and then using the midpoint of the values for the next approximation.

Example

Find an approximate negative solution to the equation $x^3 - 12x - 7 = 0$. Give your answer to 1 DP.

Solution

Let $f(x) = x^3 - 12x - 7$, so we are trying to solve $f(x) = 0$.

$f(0) = -7$ and $f(-1) = 4 \Rightarrow$ the root lies between 0 and -1

$f(-0.5) = -1.125 \Rightarrow$ the root lies between -0.5 and -1

$f(-0.75)$=1.578125 $\Rightarrow$ the root lies between -0.5 and -0.75

$f(-0.625) = 0.255859 \Rightarrow$ the root lies between -0.5 and -0.625

$f(-0.5625) = -0.42798 \Rightarrow$ the root lies between -0.5625 and -0.625

Therefore the answer is –0.6.

Note that we chose to start with $x = 0$ and $x = 1$ but we could have started with any pair of values and still have obtained the same solution.

Question 1.1

Using the bisection method, find *i* to 3DP if *i* satisfies the equation:

$$25(1+i)^{-3} + \frac{20\left(1-(1+i)^{-2}\right)}{i} = 61.5$$

Newton-Raphson iteration

The Newton-Raphson iterative formula states that if x_n is an approximate solution to the equation $f(x)=0$ then a better approximation is $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$.

Example

Using the Newton-Raphson formula, find an approximate positive solution to the equation $x^5+4x^2=7$, giving your answer to 2 DP.

Solution

Let $f(x)=x^5+4x^2-7$, so that:

$$f'(x)=5x^4+8x$$

$$f(1)=-2$$

$$f(2)=41$$

Therefore there is a solution between 1 and 2. Try $x_1=1$.

$$x_2=1.15385$$

$$x_3=1.13336$$

$$x_4=1.13290$$

Therefore the solution is 1.13 (to 2DP).

Note that we chose to start with $x=1$ *and* $x=2$ *but we could have started with any pair of values on either side of the true answer.*

Question 1.2

Using the Newton-Raphson formula, find an approximate solution to the equation:

$$5(1+i)^{-4}+\frac{10(1-(1+i)^{-5})}{i}=41$$

Give your answer to 3 DP.

Solutions

Solution 1.1

When $i = 0.03$ the LHS gives 61.15, and when $i = 0.02$ the LHS gives 62.39. Using the bisection method, we get:

Value of *i*	Value of equation
0.025	61.76
0.0275	61.45
0.02625	61.61
0.026875	61.53

So the value of *i* is 0.027 to 3DP.

We started with $i = 0.03$ and $i = 0.04$ but we could have started with any pair of values which gave answers either side of 62.

Solution 1.2

Let $f(i) = 5(1+i)^{-4} + \dfrac{10(1-(1+i)^{-5})}{i} - 41$, so that:

$$f'(i) = -20(1+i)^{-5} + \frac{i \times 10 \times 5(1+i)^{-6} - 10(1-(1+i)^{-5})}{i^2}$$

$$= -20(1+i)^{-5} + \frac{50i(1+i)^{-6} - 10 + 10(1+i)^{-5}}{i^2}$$

The Newton-Raphson formula gives the formula for the next approximation to be:

$$i - \frac{5(1+i)^{-4} + \dfrac{10(1-(1+i)^{-5})}{i} - 41}{-20(1+i)^{-5} + \dfrac{50i(1+i)^{-6} - 10 + 10(1+i)^{-5}}{i^2}}$$

$f(0.1) = 0.323$ and $f(0.11) = -0.747$, so the root lies between 0.1 and 0.11.

Try $i_1 = 0.1$.

The formula gives $i_2 = 0.1029557$, $i_3 = 0.1029739$, *ie* the root is 0.103 to 3 DP.

We started with $i = 0.1$ but we could have started with any value close to the solution.

